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The Vertex Formulation of the Bazhanov–Baxter Model

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In this paper we formulate an integrable model on the simple cubic lattice. The N-valued spin variables of the model belong to edges of the lattice. The Boltzmann weights of the model obey the vertex-type tetrahedron equation. In the thermodynamic limit our model is equivalent to the Bazhanov–Baxter model. In the case when $N=2$ we reproduce Korepanov's and Hietarinta's solutions of the tetrahedron equation as special cases.

KEY WORDS: Tetrahedron equation; Zamolodchikov model; Fermat curve; spherical geometry; symmetry properties.

1. INTRODUCTION

Recently two new solutions of the vertex type tetrahedron equation[1-3] for the number of spin states $N=2$ were obtained.[4,5] In a previous paper we generalized these solutions for $N>2$ and for general spectral parameters, and succeeded in generalizing the solution from ref. 5 for arbitrary N.

For the case $N=2$ the solution proposed by Hietarinta appears to be a special case of the planar limit of the Bazhanov–Baxter solution.[6] Recall that in the Bazhanov–Baxter model (BBM)[7] N-valued spin variables belong to the vertices of the elementary cubes of the lattice and the

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Boltzmann weights in the tetrahedron equation (TE) are parametrized by the angles of a tetrahedron.[8]

The Bazhanov–Baxter model cannot be directly reformulated as a vertex-type model using an obvious duality between vertex and interaction-round-a cube formulations. For example, for the case $N=2$ (Zamolodchikov model)[1] such a duality requires an invariance of the weight functions with respect to a recoloring of any face of the elementary cube. It is known[9] that the Boltzmann weights of the Zamolodchikov model in general do not possess this symmetry (despite the fact that the absolute values of the Boltzmann weights do).

Nevertheless, in the particular limit when all four vertices of the tetrahedron belong to the same plane (the planar limit), it is possible to rewrite the Boltzmann weights using two-state edge variables only and as a result to obtain the vertex solution of the TE from ref. 5. Note that for $N>2$ the solution of the TE from ref. 6 does not coincide with the planar limit of the BBM and seems to be new.

Attempts to remove the planar limit restriction for this solution have been unsuccessful. Instead we have obtained a complete (depending on three arbitrary angles) vertex solution of the TE for a general number of spin variables N. This solution at $N=2$ reproduces the solutions of Korepanov and Hietarinta in the static and planar limits, respectively. This new model in the thermodynamic limit coincides with the BBM. However, due to the vertex form, this formulation may be useful for a more careful investigation of the model. Namely, one can try to formulate the Bethe ansatz, and construct a functional equation for the transfer matrices analogously to the two-dimensional case. Also one can trye to construct a three-dimensional generalization of the L operators, etc.

The paper is organized as follows. In Section 2 we recall the usual notations for the functions on Z_N which will be used for constructing the Boltzmann weights. In Section 3 we give an explicit form of the vertex weight function and show the equivalence of our vertex model with the BBM in the thermodynamic limit. Symmetry properties of the vertex weight are listed in Section 4. Also we give exotic forms of the gauges and write out the inversion relation for the weight functions. The case $N=2$ is considered in a special gauge in Section 5, where we show the equivalence of our vertex weight in the static limit with the solution of the TE proposed by Korepanov. Section 6 is devoted to a sketch of the proof of the TE for the vertex weight. In an Appendix we collect the most useful formulas for ω-hypergeometric series with ω being an Nth root of unity.

2. NOTATIONS AND DEFINITIONS

In this section we give all necessary definitions and notations.
Denote

$$\omega^{1/2} = \exp(\pi i/N), \qquad N \in Z \tag{2.1}$$

Let x, y, z be three homogeneous complex variables constrained by the Fermat equation

$$x^N + y^N = z^N \tag{2.2}$$

Hereafter we use the notation $p = (x, y, z)$ unless it may lead to misunderstanding.

Now we define a function $w(p \mid a)$ by the recurrence relation

$$\frac{w(p \mid a)}{w(p \mid 0)} = \prod_{s=1}^{a} \frac{y}{z - x\omega^s} \tag{2.3}$$

where a is an element of Z_N.

Following ref. 10, we choose a normalization factor $w(p \mid 0)$ as follows. First we set $z = 1$ and consider the case $|x| < 1$. Then we can choose y as

$$y = (1 - x^N)^{1/N} \tag{2.4}$$

For such x, y, and z we put

$$w(p \mid 0) = y^{(1-N)/2} \prod_{j=1}^{N-1} (1 - \omega^{-j} x)^{j/N} \tag{2.5}$$

With such a normalization the function $w(p \mid a)$ satisfies

$$\prod_{a=0}^{N-1} w(p \mid a) = 1 \tag{2.6}$$

Further we can analytically continue formulas (2.4)–(2.6) over x into the whole complex plane with cuts from the points $x = \omega^n$, $n = 0, \ldots, N-1$, to infinity. For such x and y we will say that the point $p = (x, y, 1)$ belongs to the main branch Γ_0 of some covering curve Γ on which the function $w(p \mid a)$ is well defined. If we go under the cut around the point $x = \omega^n$ in the anticlockwise direction, then $w(p \mid 0)$ is multiplied by the phase factor $(-1)^{N-1} \omega^n$. Restoring the z dependence, it is easy to check that

$$w(\omega^n x, y, z \mid m) = \omega(x, y, z \mid m + n), \qquad m, n \in Z_N, \qquad p = (x, y, z) \in \Gamma_0 \tag{2.7}$$

Now let us consider a region on Γ_0 such that

$$-2\pi/N < \operatorname{Arg}(x/z) < 0 \tag{2.8}$$

and denote it as $\bar{\Gamma}_0$. It is easy to show that for the points $p \in \bar{\Gamma}_0$ we have

$$-\pi/N < \operatorname{Arg}(y/z) < \pi/N \tag{2.9}$$

Then for the given point $p = (x, y, z) \in \bar{\Gamma}_0$ define a new point $Op \in \bar{\Gamma}_0$ as

$$Op = (z, \omega^{1/2} y, \omega x) \tag{2.10}$$

Using these notations, we have the following property of the w function:

$$w(p \mid a)\, w(Op \mid -a)\, \Phi(a) \exp\left(\frac{i\pi(N^2-1)}{6N}\right) = 1, \qquad a \in Z_N, \quad p \in \bar{\Gamma}_0 \tag{2.11}$$

where

$$\Phi(a) = \omega^{a(a-N)/2} \tag{2.12}$$

In the appendix we give a set of the most useful formulas and identities for the w function.

3. THE VERTEX WEIGHTS

For a given spherical triangle with the angles θ_1, θ_2, θ_3 and corresponding linear angles (i.e., sides of the spherical triangle) a_1, a_2, a_3 define four points $p_i \in \bar{\Gamma}_0$:

$$x(p_1) = \omega^{-1/2} \exp\left(i\frac{a_3}{N}\right)\left(\frac{\sin\beta_1}{\sin\beta_2}\right)^{1/N},$$

$$y(p_1) = \exp\left(i\frac{\beta_1}{N}\right)\left(\frac{\sin a_3}{\sin\beta_2}\right)^{1/N}$$

$$x(p_2) = \omega^{-1/2} \exp\left(i\frac{a_3}{N}\right)\left(\frac{\sin\beta_2}{\sin\beta_1}\right)^{1/N},$$

$$y(p_2) = \exp\left(i\frac{\beta_2}{N}\right)\left(\frac{\sin a_3}{\sin\beta_1}\right)^{1/N}$$

$$x(p_3) = \omega^{-1} \exp\left(i\frac{a_3}{N}\right)\left(\frac{\sin\beta_3}{\sin\beta_0}\right)^{1/N},$$

$$y(p_3) = \exp\left(-i\frac{\beta_3}{N}\right)\left(\frac{\sin a_3}{\sin \beta_0}\right)^{1/N}$$

$$x(p_4) = \omega^{-1}\exp\left(i\frac{a_3}{N}\right)\left(\frac{\sin \beta_0}{\sin \beta_3}\right)^{1/N}$$

$$y(p_4) = \exp\left(-i\frac{\beta_0}{N}\right)\left(\frac{\sin a_3}{\sin \beta_3}\right)^{1/N}$$

$$z(p_i) = 1, \qquad i = 1, 2, 3, 4 \tag{3.1}$$

where the β_i are the usual linear excesses

$$\beta_0 = \pi - \frac{a_1 + a_2 + a_3}{2}, \qquad \beta_i = \pi - \beta_0 - a_i \tag{3.2}$$

Further we will consider (3.1) as a definition for the points $p_i \equiv p_i(a_1, a_2, a_3)$.

Let ρ_k, $k = 1, 2, 3$, be normalization factors:

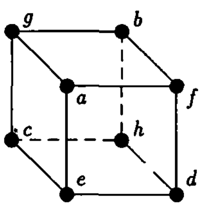
$$\rho_k = \left(\frac{\sin a_k}{2\cos(\beta_0/2)\cos(\beta_1/2)\cos(\beta_2/2)\cos(\beta_3/2)}\right)^{(N-1)/N} \tag{3.3}$$

With these notations the vertex weight function is

$$R^{j_1, j_2, j_3}_{i_1, i_2, i_3} = \delta_{j_2 + j_3, i_2 + i_3}\,\omega^{j_3(j_1 - i_1)}\rho_3\,\frac{w(p_1 \mid i_1 - i_2)\,w(p_2 \mid j_1 - j_2)}{w(p_3 \mid i_1 - j_2)\,w(p_4 \mid j_1 - i_2)} \tag{3.4}$$

Such weight functions obey the tetrahedron equation (see Section 6) and hence define an exactly solvable lattice model.

Now we will show that such a model is equivalent in the thermodynamic limit to the BBM.[7] In the BBM to each cube of the lattice we assign a weight function depending on eight corner spins (see Fig. 1).



$$= \quad W(a \mid e, f, g \mid b, c, d \mid h)$$

Fig. 1. A weight function assigned to each elementary cube of the lattice.

We choose the weight function in the following form:

$$W(a\,|\,e,f,g\,|\,b,c,d\,|\,h\,|\,a_1^B,a_2^B,a_3^B)$$
$$=\rho_3\sum_\sigma \frac{w(Op_4^B\,|\,d-h-\sigma)\,w(Op_3^B\,|\,a-g-\sigma)}{w(Op_2^B\,|\,e-c-\sigma)\,w(Op_1^B\,|\,f-b-\sigma)}\,\omega^{\sigma(h+g-c-b)} \tag{3.5}$$

where $p_i^B=p_i(a_1^B,a_2^B,a_3^B)$. Here a_i^B are the sides of the spherical triangle in the notation of refs. 7 and 8. In fact, expression (3.5) coincides with the weight function from ref. 8 up to gauge and normalization multipliers.

Consider now a chain of n weights (3.5) in the direction $a-g$ with cyclic boundary conditions. This chain defines a weight function for a two-dimensional model closely connected with the homogeneous generalized chiral Potts model.[7,12] The weight function of this two-dimensional model looks like

$$\mathscr{W}(A,B,C,D\,|\,a_1^B,a_2^B,a_3^B)$$
$$=\prod_\alpha W(a_\alpha\,|\,c_\alpha,b_\alpha,a_{\alpha+1}\,|\,b_{\alpha+1},c_{\alpha+1},d_\alpha\,|\,d_{\alpha+1})$$
$$=\sum_{\{\sigma_\alpha\}}\prod_\alpha \rho_3\frac{w(Op_4^B\,|\,\hat{d}_\alpha-\sigma_\alpha)\,w(Op_3^B\,|\,\hat{a}_\alpha-\sigma_\alpha)}{w(Op_2^B\,|\,\hat{c}_\alpha-\sigma_\alpha)\,w(Op_1^B\,|\,\hat{b}_\alpha-\sigma_\alpha)}$$
$$\times\,\omega^{\sigma_\alpha(d_{\alpha+1}+a_{\alpha+1}-c_{\alpha+1}-b_{\alpha+1})} \tag{3.6}$$

where the capital spin $A=\{a_\alpha\}$ and $\hat{a}_\alpha\equiv a_\alpha-a_{\alpha+1}$, etc. (see Fig. 2).

We now move our frame one-half step to the right in the two-dimensional lattice (see the right part of Fig. 2). The points A and C disappear from our frame, but there now appear the right neighbor Σ' of our previous spin of the summation Σ. Then we get the four-spin weight S:

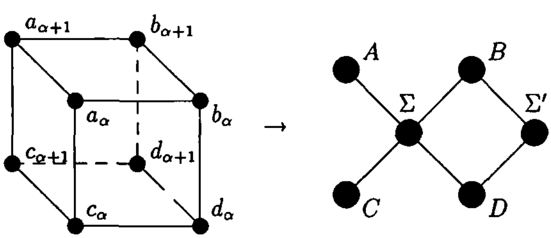


Fig. 2. A transformation corresponding to a transition from the IRC form to the vertex form of the weight functions.

$$
\begin{aligned}
S(\Sigma, B, \Sigma', D \mid a_1^B, a_2^B, a_3^B) \\
&= \prod_\alpha \rho_3 \frac{w(Op_4^B \mid \hat{d}_\alpha - \sigma_\alpha)\, w(Op_3^B \mid \hat{b}_\alpha - \sigma'_\alpha)}{w(Op_2^B \mid \hat{d}_\alpha - \sigma'_\alpha)\, w(Op_1^B \mid \hat{b}_\alpha - \sigma_\alpha)}\, \omega^{(\sigma_\alpha - \sigma'_\alpha)(d_{\alpha+1} - b_{\alpha+1})} \\
&\equiv \prod_\alpha R^{-\sigma_\alpha, -\hat{d}_\alpha, b_{\alpha+1} - d_{\alpha+1}}_{-\sigma'_\alpha, -\hat{b}_\alpha, b_\alpha - d_\alpha}(a_2^B, \pi - a_1^B, \pi - a_3^B)
\end{aligned} \tag{3.7}
$$

where the vertex weight R is defined by (3.4). The last expression in (3.7) differs from the two-dimensional projection of the vertex lattice by a slight modification of boundary conditions. So the BBM with the weight functions (3.5) depending on a_i^B is equivalent to the vertex model with the weight functions (3.4) depending on the a_i, such that

$$a_1 = a_2^B, \qquad a_2 = \pi - a_1^B, \qquad a_3 = \pi - a_3^B \tag{3.8}$$

in the thermodynamic limit.

4. SYMMETRY PROPERTIES

The weight function of the BBM, (3.5), is symmetric with respect to the cubic symmetry group up to some multiplicative gauge transformations. In the case of the vertex weight function (3.4) the corresponding gauge transformations are the Fourier ones. To simplify all formulas, we will use convenient operator notations. We will consider the weight (3.4) as an operator acting in the tensor product of three linear N-dimensional spaces so that

$$R^{j_1, j_2, j_3}_{i_1, i_2, i_3} = \langle i_1, i_2, i_3 \mid R \mid j_1, j_2, j_3 \rangle \tag{4.1}$$

Define operators of the Fourier transformation and of the spin inversion:

$$\langle i \mid F \mid j \rangle = \frac{1}{\sqrt{N}}\, \omega^{ij}, \qquad \langle i \mid J \mid j \rangle = \delta_{i, -j} \tag{4.2}$$

Then the inversion relation for the vertex weight is

$$
\begin{aligned}
R(a_1, a_2, a_3)\, J \otimes J \otimes J R(-a_1, -a_2, -a_3)\, J \otimes J \otimes J \\
= 1 \otimes 1 \otimes 1 \Phi(a_1, a_2, a_3)
\end{aligned} \tag{4.3}
$$

where

$$\Phi(a_1, a_2, a_3) = \left(\frac{\sin(\beta_0/2)}{\cos(\beta_1/2)\cos(\beta_2/2)\cos(\beta_3/2)} \right)^{2(N-1)/N} \tag{4.4}$$

In fact, expression (4.4) coincides explicitly with the inversion factor for BBM.[10]

To write down the symmetry properties of the weight (3.4) we need to define the permutation operators P_{ij}:

$$P_{12}\,|i_1, i_2, i_3\rangle = |i_2, i_1, i_3\rangle \tag{4.5}$$

and similarly for P_{13}, P_{23}.

Let t_1, t_2, t_3 be transpositions in the corresponding vector spaces, with t the total transposition sign:

$$R^t(a_1, a_2, a_3) = R^{t_1 t_2 t_3}(a_1, a_2, a_3) \tag{4.6}$$

Then the crossing relation is

$$1 \otimes 1 \otimes J R^{t_1 t_2}(a_1, a_2, a_3)\, 1 \otimes 1 \otimes J = R(\pi - a_1, \pi - a_2, a_3) \tag{4.7}$$

and all five space permutations are given by

$$\begin{gathered}
F^{-1} \otimes F^{-1} \otimes F^{-1} R(a_1, a_2, a_3)\, F \otimes F \otimes F = P_{13} R^t(a_3, a_2, a_1)\, P_{13} \\
J \otimes J \otimes F R(a_1, a_2, a_3)\, J \otimes J \otimes F^{-1} = P_{12} R^t(a_2, a_1, a_3)\, P_{12} \\
F \otimes 1 \otimes 1 R(a_1, a_2, a_3)\, F^{-1} \otimes 1 \otimes 1 = P_{23} R^t(a_1, a_3, a_2)\, P_{23} \\
F^{-1} \otimes F^{-1} \otimes J R(a_1, a_2, a_3)\, F \otimes F \otimes J = P_{23} P_{12} R(a_3, a_1, a_2)\, P_{12} P_{23} \\
J \otimes F \otimes F R(a_1, a_2, a_3)\, J \otimes F^{-1} \otimes F^{-1} = P_{12} P_{23} R(a_2, a_3, a_1)\, P_{23} P_{12}
\end{gathered} \tag{4.8}$$

Combining Fourier transformations with diagonal gauge transformations, one can obtain other forms of the R-matrix. In general these combined transformations are the gauge transformations of the lattice, but not the gauge transformations of the tetrahedron equation.

Note that there exists the conservation law $i_2 + i_3 = j_2 + j_3$ for the weight (3.4). The following are some combined transformations which lead to other forms of the spin conservations laws:

$$\begin{aligned}
N^{-1} \sum_{\alpha_3, \beta_3} \omega^{\alpha_3 j_3 - \beta_3 i_3} & \left(\frac{\Phi(\alpha_3)}{\Phi(\beta_3)}\right)^{\varepsilon} R^{j_1, j_2, \alpha_3}_{i_1, i_2, \beta_3}(a_1, a_2, a_3) \\
&= \delta(j_3 + j_1 - i_3 - i_1 - \varepsilon(j_2 - i_2))\, \Phi(j_2 - i_2)^{-\varepsilon}\, \omega^{-i_3(j_2 - i_2)} \\
&\quad \times \rho_3 \frac{w(p_1(a_1, a_2, a_3) \,|\, i_1 - i_2)\, w(p_2(a_1, a_2, a_3) \,|\, j_1 - j_2)}{w(p_3(a_1, a_2, a_3) \,|\, i_1 - j_2)\, w(p_4(a_1, a_2, a_3) \,|\, j_1 - i_2)}
\end{aligned} \tag{4.9}$$

$$N^{-2} \sum_{\alpha_m, \beta_m} \omega^{\alpha_1 j_1 - \beta_1 i_1 + \alpha_2 j_2 - \beta_2 i_2} \left(\frac{\Phi(\alpha_2)}{\Phi(\beta_2)}\right)^{\varepsilon} R^{\alpha_1, \alpha_2, j_3}_{\beta_1, \beta_2, i_3}(a_1, a_2, a_3)$$
$$= \delta(j_2 + j_1 - i_2 - i_1 - \varepsilon(j_3 - i_3))\, \Phi(j_3 - i_3)^{\varepsilon}\, \omega^{j_2(i_3 - j_3)}$$
$$\times \rho_2 \frac{w(p_1(a_1, a_3, a_2) \mid -j_1 - j_3)\, w(p_2(a_1, a_3, a_2) \mid -i_1 - i_3)}{w(p_3(a_1, a_3, a_2) \mid -j_1 - i_3)\, w(p_4(a_1, a_3, a_2) \mid -i_1 - j_3)} \tag{4.10}$$

$$N^{-3} \sum_{\alpha_m, \beta_m} \left(\prod_m \omega^{\alpha_m j_m - \beta_m i_m}\right) \left(\frac{\Phi(\alpha_1)}{\Phi(\beta_1)}\right)^{\varepsilon} R^{\alpha_1, \alpha_2, \alpha_3}_{\beta_1, \beta_2, \beta_3}(a_1, a_2, a_3)$$
$$= \delta(j_1 + j_2 - i_1 - i_2 - \varepsilon(j_3 - i_3))\, \Phi(j_3 - i_3)^{-\varepsilon}\, \omega^{i_1(i_3 - j_3)}$$
$$\times \rho_1 \frac{w(p_1(a_3, a_2, a_1) \mid j_3 - j_2)\, w(p_2(a_3, a_2, a_1) \mid i_3 - i_2)}{w(p_3(a_3, a_2, a_1) \mid j_3 - i_2)\, w(p_4(a_3, a_2, a_1) \mid i_3 - j_2)} \tag{4.11}$$

where $\delta(a) = \delta_{a,0}$, $\Phi(a)$ and ρ_k are defined by (2.12) and (3.3), respectively, and ε is an arbitrary integer. Other choices of the diagonal Φ factors lead to complicated nonmultiplicative expressions for the weights. The exception is the case $N = 2$.

5. THE CASE $N = 2$

In this section we consider the case $N = 2$ in a special gauge in which our R-matrix (3.4) can be reduced to the vertex solutions of Korepanov and Hietarinta in the static and planar limits, respectively.

For the case $N = 2$ the list of suitable Fourier transformations increases. Namely,

$$N^{-3}\xi \sum_{\alpha_m, \beta_m} \left(\prod_{m=1}^{3} \left[\omega^{\alpha_m j_m - \beta_m i_m} \frac{\Phi(\beta_m)}{\Phi(\alpha_m)}\right]\right) R^{\alpha_1, \alpha_2, \alpha_3}_{\beta_1, \beta_2, \beta_3}(a_1, a_2, a_3)$$
$$= \delta(i_1 + j_1 + i_3 + j_3 - i_2 - j_2)\, \omega^{(i_3 - j_3)(i_2 - j_3)}$$
$$\times \frac{w(r_1 \mid -i_1 + j_2 - j_3)\, w(r_3 \mid i_1 - i_2 + j_3)}{w(Or_0 \mid i_1 - i_2 + i_3)\, w(Or_2 \mid -i_1 + j_2 - i_3)} \equiv \bar{R}^{j_1, j_2, j_3}_{i_1, i_2, i_3} \tag{5.1}$$

where $\omega = -1$,

$$\xi = \{4[\cot(\beta_0/2) \cdots \cot(\beta_3/2)]^{1/2}\}^{(N-1)/N} \tag{5.2}$$

and the four points r_i are given by

$$x(r_i) = \exp(-i\beta_i/N), \qquad y(r_i) = \omega^{1/4}(2\sin\beta_i)^{1/N}$$
$$z(r_i) = \exp(i\beta_i/N), \qquad i = 0, 1, 2, 3 \tag{5.3}$$

Due to the total symmetry, this transformation is the gauge transformation of the tetrahedron equation, so this weight $\bar{R}$ obeys the tetrahedron equation (see Section 6).

When $N=2$ the function w is very simple:

$$\frac{w(r_i\,|\,1)}{w(r_i\,|\,0)}=\exp\left(i\frac{\pi}{4}\right)\left(\tan\frac{\beta_i}{2}\right)^{1/2} \tag{5.4}$$

Define

$$\left(\tan\frac{\beta_i}{2}\right)^{1/2}=t_i, \qquad i=0,\,1,\,2,\,3 \tag{5.5}$$

Then we can represent the weights (5.1) by a compact table (see Table I).

The weight defined by this table differs from (5.1) by a normalization factor. Using the following property of a spherical triangle:

$$\tan\frac{\beta_i}{2}\tan\frac{\beta_j}{2}=\tan\frac{\alpha_k}{2}\tan\frac{\alpha_l}{2}, \qquad \{i, j, k, l\}=\{0,\,1,\,2,\,3\} \tag{5.6}$$

where α_i are the angle excesses of the spherical triangle, we can easily obtain the static limit of $\bar{R}$ (the case when $\alpha_0=0$). This static limit appears to be the solution of the tetrahedron equation proposed by Korepanov.[4] Moreover, in the planar limit when $\beta_2=0$ the vertex weight (5.1) coincides with the $N=2$ solution by Hietarinta.[5,6]

Table I. Vertex Weights for the Case $N=2$

$$\begin{aligned}
&\bar{R}^{0,0,0}_{0,0,0}=\bar{R}^{0,1,1}_{0,1,1}=\bar{R}^{1,0,1}_{1,0,1}=\bar{R}^{1,1,0}_{1,1,0}=1\\
&\bar{R}^{1,1,1}_{1,1,1}=\bar{R}^{1,0,0}_{1,0,0}=\bar{R}^{0,1,0}_{0,1,0}=\bar{R}^{0,0,1}_{0,0,1}=t_0t_1t_2t_3\\[1ex]
&\bar{R}^{0,1,0}_{0,0,1}=\bar{R}^{0,0,1}_{0,1,0}=-\bar{R}^{1,0,0}_{1,1,1}=-\bar{R}^{1,1,1}_{1,0,0}=t_0t_1\\
&\bar{R}^{1,0,1}_{1,1,0}=\bar{R}^{1,1,0}_{1,0,1}=-\bar{R}^{0,1,1}_{0,0,0}=-\bar{R}^{0,0,0}_{0,1,1}=t_2t_3\\[1ex]
&\bar{R}^{1,1,1}_{0,1,0}=\bar{R}^{1,0,0}_{0,0,1}=-\bar{R}^{0,0,1}_{1,0,0}=-\bar{R}^{0,1,0}_{1,1,1}=-it_0t_2\\
&\bar{R}^{0,0,0}_{1,0,1}=\bar{R}^{0,1,1}_{1,1,0}=-\bar{R}^{1,1,0}_{0,1,1}=-\bar{R}^{1,0,1}_{0,0,0}=it_1t_3\\[1ex]
&\bar{R}^{0,0,1}_{1,1,1}=\bar{R}^{1,1,1}_{0,0,1}=\bar{R}^{1,0,0}_{0,1,0}=\bar{R}^{0,1,0}_{1,0,0}=t_0t_3\\
&\bar{R}^{1,1,0}_{0,0,0}=\bar{R}^{0,0,0}_{1,1,0}=\bar{R}^{0,1,1}_{1,0,1}=\bar{R}^{1,0,1}_{0,1,1}=t_1t_2
\end{aligned}$$

6. THE TETRAHEDRON EQUATION

The vertex form of the TE is

$$\sum_{\substack{k_1,k_2,k_3\\k_4,k_5,k_6}} R^{k_1,k_2,k_3}_{i_1,i_2,i_3} R'^{j_1k_4k_5}_{k_1i_4i_5} R''^{j_2j_4k_6}_{k_2k_4i_6} R'''^{j_3j_5j_6}_{k_3k_5k_6}$$

$$= \sum_{\substack{k_1,k_2,k_3\\k_4,k_5,k_6}} R'''^{k_3,k_5,k_6}_{i_3,i_5,i_6} R''^{k_2k_4j_6}_{i_2i_4k_6} R'^{k_1j_4j_5}_{i_1k_4k_5} R^{j_1j_2j_3}_{k_1k_2k_3} \tag{6.1}$$

A complete solution of this equation is parametrized by six angles of a tetrahedron (five of them are independent):

$$\begin{aligned} R &= R(\theta_1, \theta_2, \theta_3) \\ R' &= R(\theta_1, \theta_4, \theta_5) \\ R'' &= R(\pi-\theta_2, \theta_4, \theta_6) \\ R''' &= R(\theta_3, \pi-\theta_5, \theta_6) \end{aligned} \tag{6.2}$$

The ordering of the dihedral angles is natural with respect to the numbering of the spaces and differs from that in the standard equation (2.2) in ref. 9.

For each vertex in (6.1) let a_i be the corresponding planar angles:

$$\begin{aligned} (\theta_1, \theta_2, \theta_3) &\to (a_1, a_2, a_3) \\ (\theta_1, \theta_4, \theta_5) &\to (a'_1, a'_2, a'_3) \\ (\pi-\theta_2, \theta_4, \theta_6) &\to (a''_1, a''_2, a''_3) \\ (\theta_3, \pi-\theta_5, \theta_6) &\to (a'''_1, a'''_2, a'''_3) \end{aligned} \tag{6.3}$$

Then the planar angles of the four weights are constrained as follows:

$$\begin{aligned} a''_3 &= a'_3 - a_3, \qquad & a'''_1 &= a''_1 - a'_1 \\ a'''_2 &= a''_2 - a_1, \qquad & a'''_3 &= a'_2 - a_2 \end{aligned} \tag{6.4}$$

The four vertex weights (3.4) with the angles defined by (6.3) and satisfying (6.4) obey the tetrahedron equation (6.1). In this section sketch the proof of this statement.

Let us substitute (3.4) in (6.1). Due to the spin conservation laws, the six summations on both sides reduce to three. It is useful to choose the indices k_1, k_2, k_4 as independent spins of the summations. The summands on the left- and right-hand sides are products of phase factors $\omega^{(\cdots)}$ and w functions. First let us collect on both sides all factors depending on the spin k_1. The sums over k_1 have the form ${}_2\Psi_2$ (see the appendix). As the first step let us apply to these ${}_2\Psi_2$ the $(\tau\rho)^2$ transformations [see formula (A.14) in the appendix]. As a result there appear extra w functions, depending on $k_2 - k_4$, and we demand the cancellation of these extra factors with similar w functions on the left- and right-hand sides. This gives us some algebraic constraints on the parameters of the weights.

Further summations over k_2 and k_4 become independent. Moreover, there are no phase factors depending on k_2 and k_4, and we can sum over k_2 and k_4 using the "star–square" relation [see formula (A.19) in the appendix]. Finally, sums over k_1 are both of the type ${}_4\Psi_4$ and have a similar spin structure. Imposing necessary constraints among the parameters of the weights, we come to the equality of the left- and right-hand sides of the TE.

We will not write out here the algebraic constraints coming from the cancellation of all w functions depending on $k_2 - k_4$, the "star–square" applicability conditions (A.20), and the coincidence of the final expressions. All calculations are direct but rather tedious.

As a result we obtain that all restrictions on the parameters of the weight functions are satisfied automatically if we take into account the parametrization (3.1) and constraints between angles (6.3)–(6.4).

APPENDIX

In this appendix we collect useful formulas in the theory of ω-hypergeometric series with ω being an Nth root of unity. In fact, these formulas (or their particular cases) have appeared in many papers devoted to the chiral Potts model[14–16] and to the TE. Let us define the ${}_r\Psi_r$ series as

$$\begin{aligned}
&{}_r\Psi_r\left(\begin{matrix}(p_1, m_1)\cdots(p_r, m_r)\\(p_1', m_1')\cdots(p_r', m_r')\end{matrix}\,\middle|\, n\right)\\
&\qquad = \sum_{\sigma\in Z_N} \frac{w(p_1\,|\,m_1+\sigma)\cdots w(p_r\,|\,m_r+\sigma)}{w(p_1'\,|\,m_1'+\sigma)\cdots w(p_r'\,|\,m_r'+\sigma)}\,\frac{\omega^{n\sigma}}{\sqrt{N}}
\end{aligned}\tag{A.1}$$

Let us now discuss the role of the normalization. Spin-independent factors in all identities in this appendix are given for the case when all arguments of the w functions on the left- and right-hand sides belong to the region $\bar{\Gamma}_0$ [see (2.8)]. If we abandon the restriction (2.8), then the phases of the w's can be chosen in such a way that the corresponding formulas remain correct.

To simplify all notations, we omit arguments in the components of points p_i and imply that

$$p_i = (x_i, y_i, z_i) \tag{A.2}$$

for every i. In the final formulas of this appendix we will use also points q_i. In this case we will explicitly point out by a superscript the corresponding point

$$q_i = (x_i^q, y_i^q, z_i^q) \tag{A.3}$$

Many new points will appear on the Fermat curve on the right-hand sides of the formulas. In these cases we have to introduce new letters for y components. They have to be defined by (2.4) [and belong to region (2.9) in accordance with our agreement].

All formulas in this appendix are summation formulas (which exist for $r = 1, 2, 3$) and symmetry transformation formulas (which exist for $r = 1, 2, 3, 4$).

We begin with a cyclic analog of the Ramanujan summation formula for $r = 1$. In fact, this is nothing but the restricted star–triangle relation of the Bazhanov–Baxter model[13]

$$\begin{aligned}
&{}_1\Psi_1\left(\begin{matrix}(p_1, m_1)\\(p_2, m_2)\end{matrix}\,\middle|\, n\right)\\
&\quad = \Phi_0\left(\frac{\xi}{y_1 y_2}\right)^{(N-1)/2} \frac{w(z_1 y_2, \xi, y_1 z_2 \,|\, -n)\; w(x_1 z_2, \xi, \omega z_1 x_2 \,|\, m_1 - m_2)}{\omega^{nm_2} w(x_1 y_2, \xi, \omega y_1 x_2 \,|\, m_1 - m_2 - n)}\\
&\quad = \left(\frac{\omega^{-1/2}\xi}{y_1 y_2}\right)^{(N-1)/2}\\
&\qquad \times \frac{\Phi_0^{-1}\omega^{-nm_1} w(y_1 x_2, \omega^{-1/2}\xi, x_1 y_2 \,|\, m_2 - m_1 + n)}{w(y_1 z_2/\omega, \omega^{-1/2}\xi, z_1 y_2 \,|\, n)\; w(z_1 x_2, \omega^{-1/2}\xi, x_1 z_2 \,|\, m_2 - m_1)}
\end{aligned} \tag{A.4}$$

where

$$\Phi_0 = \exp\left\{i\pi \frac{(N-1)(N-2)}{12N}\right\} \tag{A.5}$$

For the proof of this formula see, for example, refs. 11 and 13.

Here we give also two forms of inversion relations for w functions which have been used for the proof of the inversion relation (4.3):

$$\sum_\sigma \frac{w(\omega x, \omega y, z \,|\, m_1 + \sigma)}{w(x, y, z \,|\, m_2 + \sigma)} = N\delta_{m_1, m_2}\left(\frac{\omega^{1/2} x}{y}\right)^{N-1} \tag{A.6}$$

$$\sum_\sigma \frac{w(x, y, z \,|\, m_1 + \sigma)}{w(x, y, \omega z \,|\, m_2 + \sigma)} = N\delta_{m_1, m_2}\left(\frac{\omega^{1/2} x}{y}\right)^{N-1} \tag{A.7}$$

Note that the points $(\omega x, \omega y, z)$ and $(x, y, \omega z)$ do not belong to the region $\bar{\Gamma}_0$ and we define for these cases

$$\frac{w(\omega x, \omega y, z\,|\,0)}{w(x, y, z\,|\,0)} = -\omega^{1/2}\frac{y}{z-\omega x} \tag{A.8}$$

$$\frac{w(x, y, \omega z\,|\,0)}{w(x, y, z\,|\,0)} = -\omega^{1/2}\frac{z-x}{y} \tag{A.9}$$

where $(x, y, z) \in \bar{\Gamma}_0$.

To obtain symmetry formulas for higher r, we use the following simple fact. Let g_1 and g_2 be arbitrary functions on Z_N. If

$$\tilde{g}_i(k) = \sum_{\sigma \in Z_N} g_i(\sigma)\frac{\omega^{k\sigma}}{\sqrt{N}} \tag{A.10}$$

then

$$\sum_{\sigma \in Z_N} g_1(\sigma)\, g_2(\sigma) = \sum_{\sigma \in Z_N} \tilde{g}_1(\sigma)\, \tilde{g}_2(-\sigma) \tag{A.11}$$

Using this, it is easy to obtain the following symmetry transformation for ${}_2\Psi_2$:

$$\begin{aligned}
&{}_2\Psi_2\left(\begin{matrix}(p_1, m_1)(p_3, m_3)\\(p_2, m_2)(p_4, m_4)\end{matrix}\,\middle|\, n\right)\\
&\quad = {}_2\Psi_2\left(\begin{matrix}(q_1, 0) & (q_3, m_4-m_3+n)\\(q_2, n) & (q_4, m_1-m_2)\end{matrix}\,\middle|\, m_2-m_3\right)\\
&\qquad \times\left(\frac{\xi_{12}\xi_{43}}{y_1\, y_2\, y_3\, y_4}\right)^{(N-1)/2}\omega^{-nm_3}\frac{w(x_1z_2, \xi_{12}, \omega z_1x_2\,|\,m_1-m_2)}{w(z_3x_4, \xi_{43}, x_3z_4\,|\,m_4-m_3)}
\end{aligned} \tag{A.12}$$

where

$$\begin{aligned}
q_1 &= (z_1\, y_2, \xi_{12}, y_1 z_2), \qquad & q_3 &= (y_3x_4, \xi_{43}, x_3\, y_4)\\
q_2 &= (y_3z_4/\omega, \xi_{43}, z_3\, y_4), \qquad & q_4 &= (x_1\, y_2, \xi_{12}, \omega y_1 x_2)
\end{aligned} \tag{A.13}$$

This relation appeared originally as the $(\tau\rho)$ transformation in ref. 11 for the BB weight function. Note that $(\tau\rho)^6 = 1$. In this paper we have used the $(\tau\rho)^2$ transformation:

$$
\begin{aligned}
&{}_2\Psi_2\left(\begin{matrix}(p_1, m_1)(p_3, m_3)\\(p_2, m_2)(p_4, m_4)\end{matrix}\,\middle|\, n\right)\\
&\quad = {}_2\Psi_2\left(\begin{matrix}(s_1, 0) & (s_3, m_1 - m_4 - n)\\(s_2, m_2 - m_3) & (s_4, -n)\end{matrix}\,\middle|\, m_3 - m_4\right)\\
&\quad\times \omega^{-nm_2 - (m_2 - m_3)(m_4 - m_3)}\left(\frac{\Lambda\Lambda'}{\xi_{12}\xi_{43}\, y_1 y_2 y_3 y_4}\right)^{(N-1)/2}\\
&\quad\times \frac{w(x_1 z_2, \xi_{12}, \omega z_1 x_2 \,|\, m_1 - m_2)}{w(z_3 x_4, \xi_{43}, x_3 z_4 \,|\, m_4 - m_3)}\,\frac{w(z_1 z_3 y_2 y_4, \Lambda, z_2 z_4 y_1 y_3 \,|\, -n)}{w(x_1 x_3 y_2 y_4, \Lambda', \omega x_2 x_4 y_1 y_3 \,|\, -\bar{n})}
\end{aligned}
\tag{A.14}
$$

where

$$
\bar{n} = n - m_1 - m_3 + m_2 + m_4 \tag{A.15}
$$

and

$$
\begin{aligned}
s_1 &= (y_1 z_2 \xi_{43}, \Lambda, z_3 y_4 \xi_{12}), & s_3 &= (x_1 y_2 \xi_{43}, \Lambda', y_3 x_4 \xi_{12})\\
s_2 &= (y_1 x_2 \xi_{43}, \Lambda', x_3 y_4 \xi_{12}), & s_4 &= (z_1 y_2 \xi_{43}, \Lambda, y_3 z_4 \xi_{12})
\end{aligned}
\tag{A.16}
$$

We finish the list of the symmetry formulas for ${}_2\Psi_2$ by $T = (\tau p)^3$:

$$
\begin{aligned}
&{}_2\Psi_2\left(\begin{matrix}(p_1, m_1)(p_3, m_3)\\(p_2, m_2)(p_4, m_4)\end{matrix}\,\middle|\, n\right)\\
&\quad = {}_2\Psi_2\left(\begin{matrix}(\bar{p}_1, -m_3)(\bar{p}_3, -m_1)\\(\bar{p}_2, -m_4)(\bar{p}_4, -m_2)\end{matrix}\,\middle|\, \bar{n}\right)\omega^{-nm_2 - m_4\bar{n}}\left(\frac{\xi_{12}\xi_{43}\xi_{32}\xi_{41}}{\Lambda\Lambda'}\right)^{(N-1)/2}\\
&\quad\times \frac{w(x_1 z_2, \xi_{12}, \omega z_1 x_2 \,|\, m_1 - m_2)}{w(z_3 x_4, \xi_{43}, x_3 z_4 \,|\, m_4 - m_3)}\,\frac{w(z_1 z_3 y_2 y_4, \Lambda, z_2 z_4 y_1 y_3 \,|\, -n)}{w(x_1 x_3 y_2 y_4, \Lambda', \omega x_2 x_4 y_1 y_3 \,|\, -\bar{n})}\\
&\quad\times \frac{w(x_3 z_2, \xi_{32}, \omega z_3 x_2 \,|\, m_3 - m_2)}{w(z_1 x_4, \xi_{41}, x_1 z_4 \,|\, m_4 - m_1)}
\end{aligned}
\tag{A.17}
$$

where

$$
\begin{aligned}
\bar{p}_1 &= (z_3 \Lambda', \xi_{32}\xi_{43} y_1, x_3 \Lambda), & \bar{p}_3 &= (z_1 \Lambda', \xi_{41}\xi_{12} y_3, x_1 \Lambda)\\
\bar{p}_2 &= (z_4 \Lambda', \omega\xi_{41}\xi_{43} y_2, \omega x_4 \Lambda), & \bar{p}_4 &= (z_2 \Lambda', \xi_{32}\xi_{12} y_4, \omega x_2 \Lambda)
\end{aligned}
\tag{A.18}
$$

Note that in the case when $x_1 x_3 / x_2 x_4 = \omega(z_1 z_3 / z_2 z_4)$, (A.17) becomes the star–star relation for the BBM, and $p_i = \bar{p}_i$.

To obtain a summation formula for ${}_2\Psi_2$, consider (A.12) and set $n = 0$ and $q_1 = q_2$. Then applying (A.4) to the right-hand side of (A.12), we obtain the "star–square" relation:

$$
{}_2\Psi_2\left(\left.\frac{(p_1,m_1)(p_3,m_3)}{(p_2,m_2)(p_4,m_4)}\right|0\right)
= \left(\frac{\omega^{1/2}\Lambda'}{y_1 y_2 y_3 y_4}\right)^{(N-1)/2} \omega^{-(m_2-m_3)(m_1-m_2)}
$$
$$
\times \Phi_0 w(x_2 x_4 y_1 y_3, \omega^{-1/2}\Lambda', x_1 x_3 y_2 y_4 \mid m_2+m_4-m_1-m_3)
$$
$$
\times \frac{w(x_1 z_2, \xi_{12}, \omega z_1 x_2 \mid m_1-m_2)}{w(z_3 x_4, \xi_{43}, x_3 z_4 \mid m_4-m_3)} \frac{w(x_3 z_2, \xi_{32}, \omega z_3 x_2 \mid m_3-m_2)}{w(z_1 x_4, \xi_{41}, x_1 z_4 \mid m_4-m_1)} \tag{A.19}
$$

where the parameters on the left-hand side obey the special restriction

$$
\frac{y_1 y_3}{y_2 y_4} = \omega \frac{z_1 z_3}{z_2 z_4} \tag{A.20}
$$

and the phases on the right-hand side are constrained by

$$
\frac{\xi_{12}}{\xi_{43}} = \frac{y_1 z_2}{y_4 z_3}, \qquad \frac{\xi_{32}}{\xi_{41}} = \frac{y_3 z_2}{y_4 z_1}, \qquad \Lambda' = \omega^{-1/2}\xi_{12}\xi_{32}\frac{y_4}{z_2} \tag{A.21}
$$

We will try to avoid such long notations as in (A.17) and (A.19). Extra w multipliers in all consequent formulas will have the same structure as on the right-hand side of (A.17) and so we will use only ξ_{ij} to denote the whole argument dependence of w.

Consider now $r=3$. A summation formula can be obtained by summing (A.17) over n with the help of the restricted star–triangle relation (A.4). The result reads

$$
{}_3\Psi_3\left(\left.\frac{(p_1,m_1)(p_3,m_3)(q_1,m_2+m_4-\lambda)}{(p_2,m_2)(p_4,m_4)(q_2,m_1+m_3-\lambda)}\right|0\right)
= \Phi_0^{-1}\left(\frac{\xi_{12}\xi_{43}\xi_{32}\xi_{41}}{y_1 y_2 y_3 y_4 \Xi}\right)^{(N-1)/2}
$$
$$
\times \frac{\omega^{(m_4-\lambda)(m_1+m_3-m_2-m_4)}}{w(x_1 x_3 z_2 z_4, \Xi, \omega^2 x_2 x_4 z_1 z_3 \mid m_1+m_3-m_2-m_4)}
$$
$$
\times \frac{w(\xi_{12} \mid m_1-m_2)\, w(\xi_{32} \mid m_3-m_2)}{w(\xi_{43} \mid m_4-m_3)\, w(\xi_{41} \mid m_4-m_1)}
$$
$$
\times \frac{w(\bar{p}_1 \mid \lambda-m_3)\, w(\bar{p}_3 \mid \lambda-m_1)}{w(\bar{p}_2 \mid \lambda-m_4)\, w(\bar{p}_4 \mid \lambda-m_2)} \tag{A.22}
$$

where $w(\xi_{ij})$ and $w(\bar{p}_i)$ are the same as in (A.17) and

$$\begin{aligned} q_1 &= (\omega x_2 x_4 \Lambda, y_2 y_4 \Xi, z_2 z_4 \Lambda') \\ q_2 &= (x_1 x_3 \Lambda, y_1 y_3 \Xi, \omega z_1 z_3 \Lambda') \end{aligned} \tag{A.23}$$

Note that the formula (A.22) is symmetric with respect to any permutation of p_1, p_3, q_1 and p_2, p_4, q_2. The star–triangle relation[15,16] for the chiral Potts model is a special case of (A.22).

To obtain symmetry relations for $r=3$ and $r=4$, we have to use (A.11), apply (A.4) or the T transformation correspondingly, cancel extra w factors (this gives some constraints), and then, using (A.11) again, obtain the corresponding ${}_r\Psi_r$ on the right-hand side. The formula for $r=3$ reads

$$\begin{aligned} &{}_3\Psi_3\left(\begin{matrix}(p_1, m_1)(p_3, m_3)(q_1, l_1) \\ (p_2, m_2)(p_4, m_4)(q_2, l_2)\end{matrix}\,\middle|\, 0\right) \\ &= \left(\frac{\xi_{12}\xi_{43}\xi_{32}\xi_{41}\xi}{\Lambda'^2 \Lambda y_1^q y_2^q}\right)^{(N-1)/2} \omega^{(l_1 - m_2)(m_1 + m_3 - m_2 - m_4)} \\ &\times \frac{w(\xi_{12} \mid m_1 - m_2)\, w(\xi_{32} \mid m_3 - m_2)}{w(\xi_{43} \mid m_4 - m_3)\, w(\xi_{41} \mid m_4 - m_1)} \, \frac{w(\xi \mid l_2 + m_2 + m_4 - l_1 - m_1 - m_3)}{w(\lambda \mid l_2 - l_1)} \\ &\times {}_3\Psi_3\left(\begin{matrix}(\bar{p}_1, -m_3)(\bar{p}_3, -m_1)(\bar{q}_1, l_1 - m_2 - m_4) \\ (\bar{p}_2, -m_4)(\bar{p}_4, -m_2)(\bar{q}_2, l_2 - m_1 - m_3)\end{matrix}\,\middle|\, 0\right) \end{aligned} \tag{A.24}$$

where the connections between the arguments on the left-hand side are

$$\frac{y_1^p y_3^p y_1^q}{y_2^p y_4^p y_2^q} = \omega \frac{z_1^p z_3^p z_1^q}{z_2^p z_4^p z_2^q}, \qquad \frac{\Lambda}{\lambda} = \frac{z_2^p z_4^p y_1^p y_3^p}{z_1^q y_2^q} \tag{A.25}$$

and the new arguments on the right-hand side of (A.24) are

$$\begin{aligned} (\xi) &= (\omega x_2^p x_4^p x_2^q y_1^p y_3^p y_1^q, \xi, \omega x_1^p x_3^p x_1^q y_2^p y_4^p y_2^q) \\ (\lambda) &= (z_1^q x_2^q, \lambda, x_1^q z_2^q) \\ \bar{q}_1 &= (x_1^q y_2^q \Lambda', \xi, \omega \lambda x_2^p x_4^p y_1^p y_3^p) \\ \bar{q}_2 &= (y_1^q x_2^q \Lambda', \xi, \omega \lambda x_1^p x_3^p y_2^p y_4^p) \end{aligned} \tag{A.26}$$

Note that this formula is a symmetry transformation for something. Denote (A.24) as ρ_3. Let τ_3 be a permutation transformation, reordering the columns in ${}_3\Psi_3$ as $\tau_3(1, 2, 3) = (2, 3, 1)$. Then $(\tau_3 \rho_3)^6 = 1$.

The last formula is a symmetry transformation for ${}_4\Psi_4$. A derivation of it is described before formula (A.24). Let the structure of a set q_i, $\bar{q}_i$, χ_{ij}, Δ, Δ' be defined identically to that of p_i, $\bar{p}_i$, ξ_{ij}, Λ, Λ'. Then

$$
{}_4\Psi_4\left(\begin{matrix}(p_1,m_1)(p_3,m_3)(q_1,l_1)(q_3,l_3)\\(p_2,m_2)(p_4,m_4)(q_2,l_2)(q_4,l_4)\end{matrix}\,\middle|\,0\right)
$$
$$
=\left(\frac{\xi_{12}\xi_{43}\xi_{32}\xi_{41}\chi_{12}\chi_{43}\chi_{32}\chi_{41}}{\Lambda'\Lambda\Delta'\Delta}\right)^{(N-1)/2}\frac{\omega^{(l_2-m_2)(m_1+m_3-m_2-m_4)}}{\Phi(m_1+m_3-m_2-m_4)}
$$
$$
\times\frac{w(\xi_{12}\,|\,m_1-m_2)\,w(\xi_{32}\,|\,m_3-m_2)}{w(\xi_{43}\,|\,m_4-m_3)\,w(\xi_{41}\,|\,m_4-m_1)}\,\frac{w(\chi_{12}\,|\,l_1-l_2)\,w(\chi_{32}\,|\,l_3-l_2)}{w(\chi_{43}\,|\,l_4-l_3)\,w(\chi_{41}\,|\,l_4-l_1)}
$$
$$
\times{}_4\Psi_4\left(\frac{(\bar p_1,-m_3)(\bar p_3,-m_1)(\bar q_1,l_1-m_2-m_4)(\bar q_3,l_3-m_2-m_4)}{(\bar p_2,-m_4)(\bar p_4,-m_2)(\bar q_2,l_2-m_1-m_3)(\bar q_4,l_4-m_1-m_3)}\,\middle|\,0\right) \tag{A.27}
$$

where the constraints are

$$
\frac{y_1^p y_3^p y_1^q y_3^q}{y_2^p y_4^p y_2^q y_4^q}=\omega\,\frac{z_1^p z_3^p z_1^q z_3^q}{z_2^p z_4^p z_2^q z_4^q}=\omega^{-1}\,\frac{x_1^p x_3^p x_1^q x_3^q}{x_2^p x_4^p x_2^q x_4^q} \tag{A.28}
$$

and

$$
\frac{\Lambda}{\Delta}=\omega^{-1/2}\,\frac{y_1^p y_3^p z_2^p z_4^p}{z_1^q z_3^q y_2^q y_4^q},\qquad \frac{\Lambda'}{\Delta'}=\omega^{1/2}\,\frac{y_1^p y_3^p x_2^p x_4^p}{x_1^q x_3^q y_2^q y_4^q} \tag{A.29}
$$

and the spins in (A.27) are not independent:

$$
m_1+m_3+l_1+l_3=m_2+m_4+l_2+l_4 \tag{A.30}
$$

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